

Towards a good notion of categories of logics

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Abstract

We consider (finitary, propositional) logics through the original use of Category Theory: the study of the “sociology of mathematical objects”, aligning us with a recent, and growing, trend of study logics through its relations with other logics (e.g. process of combinations of logics as fibring **[Gab]** and possible translation semantics **[Car]**). So will be objects of study the classes *of* logics, i.e. categories whose objects are logical systems (i.e., a signature with a Tarskian consequence relation) and the morphisms are related to (some concept of) translations between these systems. The present work provides the first steps of a project of considering categories of logical systems satisfying *simultaneously* certain natural requirements: it seems that in the literature (**[AFLM1]**, **[AFLM2]**, **[AFLM3]**, **[BC]**, **[BCC1]**, **[BCC2]**, **[CG]**, **[FC]**) this is achieved only partially.

Introduction

We consider (finitary, propositional) logics through the original use of Category Theory: the study of the “sociology of mathematical objects”, aligning us with a recent, and growing, trend of study logics through its relations with other logics, e.g. in the process of combinations of logics. The phenomenon of combinations of logics (**[CC3]**), emerged in the mid-1980s, was the main motivation for considering categories of logics. There are two aspects of combination of logics: (i) splitting of logics: a analitical process; (ii) splicing of logics: a synthesis. The “Possible-Translations Semantics”, introduced in **[Car]**, is an instance of the splitting process: a given logic system is decomposed into other (simpler) systems, providing, for instance a conservative translation of the logic in analysis into a “product” (or weak product) of simpler or better known logics. The “Fibering” of logics, introduced originally in the context of modal logics (**[Gab]**), is “the least logic which extends simultaneously the given logics”; after, this was recognized as a coproduct construction (**[SSC]**): this provides an example of synthesis of logics.

In the field of categories of logics there are, of course, two choices that must be done: (i) the choice of objects (how represent a logical system?); (ii) the choice of arrows (what are the relevant notions of morphisms between logics?). Here we took very simple and universal choices: a logical system will be a (finitary) signature endowed a Tarskian consequence relation and the morphisms are related to (some concept of) “logical translations” between these systems.

The main flow of research on categories of logics, represented by the groups of CLE-Unicamp (Brazil) and IST-Lisboa (Portugal) focus on the determination of the conditions for preservation of metalogical properties under the process of combination of logics (**[Con]**, **[CCCCS]**, **[CR]**, **[SRC]**, **[ZSS]**). On the

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other hand, the "global aspects" of categories of logics, that ensure for example the abundance or scarcity of constructions, seem to have not been adequately studied.

The present work provides the first steps of a project of considering categories of logical systems satisfying *simultaneously* certain natural requirements such as:

- (i) If they represent the majority of the usual logical systems;
- (ii) If they have good categorial properties (e.g., if they are a complete and/or cocomplete category, if they are accessible categories ([AR]));
- (iii) If they allow a natural notion of *algebraizable* logical system (as in the concept of Blok-Pigozzi algebraizable logic ([BP]) or Czelakowski's proto-algebraizability ([Cze]));
- (iv) If they provide a satisfactory treatment of the *identity problem* of logical systems (when logics can be considered "the same"? ([Bez], [CG])).

In the series of articles [AFLM1], [AFLM2], [AFLM3], was considered a simple (but too strict) notion of morphism of signatures, where are founded some categories of logics that satisfy simultaneously the first three requirements, but not the item (iv); here we will denote by \mathcal{S}_s and \mathcal{L}_s the category of signatures and of logics therein.

In the series of papers [BC], [BCC1], [BCC2], [CG]¹, [FC] is developed a more flexible notion of morphism of signatures based on formulas as connectives (our notation for the associated category of signatures will be \mathcal{S}_f and \mathcal{L}_f will denote the associated category of logics), it encompasses items (i) and (iii) and allows some treatment of item (iv), but does not satisfy (ii).

In [MM] we provide an approach to overcome both the deficiencies of the two series of papers. In the present work we provide some new and more detailed information on the categories of signatures underlying to the categories of logics in the two series of papers above mentioned and also in [MM]: We present notions of categories of logical systems (and of of signatures) that do not impose too many constraints and that have not many categorial failures. We preserve the usual the notion of (finitary, propositional) logic as a pair formed by a (finitary) signature and a Tarskian consequence relation on the associated set of formulas on denumerable variables, but we change the notion of (translation) morphism between logics to allow more interesting connections between logics. The basic idea is to take quotient categories of categories of logics and translations by a (congruence) relation that identifies two morphisms if, for each formula in the domain logic, the associated formulas images by the morphisms in the codomain logic are interdemonstrable, but in fact we work with reflective subcategory of this quotient category determined by "well-behaved" logics.

We briefly describe the paper. Section 1 consider only categories of signatures: in (1.1) we recall the basic properties of the categories of signatures \mathcal{S}_s and \mathcal{S}_f and we add some new information; in (1.2) we

compare these two categories of signatures by means of functors $\mathcal{S}_s \overset{(+)}{\rightleftarrows} \mathcal{S}_f \overset{(-)}{\rightleftarrows} \mathcal{S}_s$ and we prove that they

provide an adjoint pair of functors; in (1.3) we identify the monad (or triple) associated to the described adjunction, we identify some properties of the monad, we prove that \mathcal{S}_f is, precisely, the *Kleisli category* of that monad and we extract some consequences. Section 2 deals with categories of logics: in (2.1) we describe in details the natural structure of algebraic lattice on the set of consequence relations over a given signature; in (2.2) we recall the basic properties of the categories of logics \mathcal{L}_s and \mathcal{L}_f ; in (2.3) we present new information on \mathcal{L}_f and we prove that the results on categories of signatures presented in (1.3) and (1.4) "lift" to categories of logics: this constitutes evidence that the defects of \mathcal{L}_f are all inherited from \mathcal{S}_f ; in (2.4) we introduce new categories of logics that solves the "deficiencies" of the categories of logics presented in the literature. We finish the paper in Section 3 with some comments and future perspectives.

¹We Thank professor Marcelo Coniglio for that reference.

In what follows, $X = \{x_0, x_1, \dots, x_n, \dots\}$ will denote a fixed enumerable set (written in a fixed order).

1 Categories of signatures

1.1 Known facts about categories of signatures

1.1.1 The category \mathcal{S}_s

We will write \mathcal{S}_s for the category of signatures and *strict* morphisms of signatures presented in [AFLM1], [AFLM2], [AFLM3], and described below.

The objects of \mathcal{S}_s are signatures. A signature Σ is a sequence of sets $\Sigma = (\Sigma_n)_{n \in \omega}$ such that $\Sigma_i \cap \Sigma_j = \emptyset$ for all $i < j < \omega$. We write $|\Sigma| = \bigcup_{n \in \omega} \Sigma_n$ for the *support* of Σ and we denote by $F(\Sigma)$, the *formula algebra* of Σ , i.e. the set of all (propositional) formulas built with signature Σ over the variables in X . For all $n \in \mathbb{N}$ let $F(\Sigma)[n] = \{\varphi \in F(\Sigma) : \text{var}(\varphi) = \{x_0, x_1, \dots, x_{n-1}\}\}$, where $\text{var}(\varphi)$ is the set of all variables that occur in the Σ -formula φ . The notion of complexity $\text{compl}(\varphi)$ of the formula φ is, as usual, the number of occurrences of connectives in φ .

If Σ, Σ' are signatures then a *strict* morphism $f : \Sigma \rightarrow \Sigma'$ is a sequence of functions $f = (f_n)_{n \in \omega}$, where $f_n : \Sigma_n \rightarrow \Sigma'_n$. Composition and identities in \mathcal{S}_s are componentwise.

For each morphism $f : \Sigma \rightarrow \Sigma'$ in \mathcal{S}_s there is a unique function $\hat{f} : F(\Sigma) \rightarrow F(\Sigma')$, called the *extension* of f , such that: (i) $\hat{f}(x) = x$, if $x \in X$; (ii) $\hat{f}(c_n(\psi_0, \dots, \psi_{n-1})) = f_n(c_n)(\hat{f}(\psi_0), \dots, \hat{f}(\psi_{n-1}))$, if $c_n \in \Sigma_n$. Then, by induction on the complexity of formulas:

(0) $\text{compl}(\hat{f}(\theta)) = \text{compl}(\theta)$, for all $\theta \in F(\Sigma)$.

(1) If $\text{var}(\theta) \subseteq \{x_{i_0}, \dots, x_{i_{n-1}}\}$, then $\hat{f}(\theta(\vec{x})[\vec{x} \mid \vec{\psi}]) = (\hat{f}(\theta(\vec{x}))[\vec{x} \mid \hat{f}(\vec{\psi})])$. Moreover $\text{var}(\hat{f}(\theta)) = \text{var}(\theta)$ and then \hat{f} restricts to maps $\hat{f} \upharpoonright_n : F(\Sigma)[n] \rightarrow F(\Sigma')[n]$, $n \in \mathbb{N}$.

(2) The extension to the formula algebra of a composition is the extension's composition. The extension of an identity is the identity function on the formula algebra.

Remark that \mathcal{S}_s is equivalent to the functor category $\mathbf{Set}^{\mathbb{N}}$, where \mathbb{N} is the discrete category with object class \mathbb{N} , then \mathcal{S} has all small limits and colimits and they are componentwise. Moreover, the category \mathcal{S}_s is a finitely locally presentable category, i.e., \mathcal{S}_s is a finitely accessible category that is cocomplete and/or complete([AR]). The finitely presentable signatures are precisely the signatures of finite support.

(Sub) For any *substitution* function $\sigma : X \rightarrow F(\Sigma)$, there is unique *extension* $\tilde{\sigma} : F(\Sigma) \rightarrow F(\Sigma)$ such that $\tilde{\sigma}$ is an “homomorphism”: $\tilde{\sigma}(x) = \sigma(x)$, for all $x \in X$ and $\tilde{\sigma}(c_n(\psi_0, \dots, \psi_{n-1})) = c_n(\tilde{\sigma}(\psi_0), \dots, \tilde{\sigma}(\psi_{n-1}))$, for all $c_n \in \Sigma_n$, $n \in \omega$; it follows that for any $\theta(x_0, \dots, x_{n-1}) \in F(\Sigma)$ $\tilde{\sigma}(\theta(x_0, \dots, x_{n-1})) = \theta(\sigma(x_0), \dots, \sigma(x_{n-1}))$. The *identity substitution* induces the identity homomorphism on the formula algebra; the *composition substitution* of the substitutions $\sigma', \sigma : X \rightarrow F(\Sigma)$ is the substitution $\sigma'' : X \rightarrow F(\Sigma)$, $\sigma'' = \sigma' \star \sigma := \tilde{\sigma'} \circ \sigma$ and $\widetilde{\sigma''} = \widetilde{\sigma' \star \sigma} = \tilde{\sigma'} \circ \tilde{\sigma}$.

(3) Let $f : \Sigma \rightarrow \Sigma'$ be a \mathcal{S}_s -morphism. Then for each substitution $\sigma : X \rightarrow F(\Sigma)$ there is a substitution $\sigma' : X \rightarrow F(\Sigma')$ such that $\sigma' \circ \hat{f} = \hat{f} \circ \tilde{\sigma}$.

1.1.2 The category \mathcal{S}_f

We will write \mathcal{S}_f for the category of signatures and *flexible* morphisms of signatures presented in the series of papers [BC], [BCC1], [BCC2], [CG], [FC] and described below.

We introduce the following notations:

If $\Sigma = (\Sigma_n)_{n \in \mathbb{N}}$ is a signature, then write $T(\Sigma) := (F(\Sigma)[n])_{n \in \mathbb{N}}$; clearly $T(\Sigma)$ satisfies the “disjunction

condition", then it is a signature too.

We have the inverse bijections (just notations):

$$h \in \mathcal{S}_f(\Sigma, \Sigma') \iff h^\# \in \mathcal{S}_s(\Sigma, T(\Sigma')) \iff f \in \mathcal{S}_s(\Sigma, T(\Sigma')) \iff f^\flat \in \mathcal{S}_f(\Sigma, \Sigma').$$

For each signature Σ and $n \in \mathbb{N}$, let the function:

$$(j_\Sigma)_n : \Sigma_n \longrightarrow F(\Sigma)[n] \quad : \quad c_n \mapsto c_n(x_0, \dots, x_{n-1}).$$

For each morphism $f : \Sigma \longrightarrow \Sigma'$ in \mathcal{S}_f there is a unique function $\check{f} : F(\Sigma) \longrightarrow F(\Sigma')$, called the *extension of f* , such that: (i) $\check{f}(x) = x$, if $x \in X$; (ii) $\check{f}(c_n(\psi_0, \dots, \psi_{n-1})) = (f_n(c_n)(x_0, \dots, x_{n-1}))[x_0 \mid \check{f}(\psi_0), \dots, x_{n-1} \mid \check{f}(\psi_{n-1})]$, if $c_n \in \Sigma_n$.

The notion of extension of \mathcal{S}_f -morphism to formula algebras shares many properties with notion of extension of \mathcal{S}_s -morphism to formula algebras: e.g., the properties **(1)**, **(2)**, **(3)**.

The composition in \mathcal{S}_f is given by $(f' \bullet f)^\# := (\check{f}' \upharpoonright_n \circ (f^\#)_n)_{n \in \mathbb{N}}$. The identity id_Σ in \mathcal{S}_f is given by $id_\Sigma^\# = ((j_\Sigma)_n)_{n \in \mathbb{N}}$.

Remark that the "information encoded" by the extension of \mathcal{S}_f -morphism is enough to determine that morphism. More precisely, given $g, f \in \mathcal{S}_f(\Sigma, \Sigma')$, note that:

- * $(f^\#)_n = \check{f} \upharpoonright_n \circ (j_\Sigma)_n$, $n \in \mathbb{N}$;
- * $\check{f} = \check{g} \Rightarrow f^\# = (\check{f} \upharpoonright_n)_{n \in \mathbb{N}} \circ j_\Sigma = (\check{g} \upharpoonright_n)_{n \in \mathbb{N}} \circ j_\Sigma = g^\# \Rightarrow f = g$.

For the reader's convenience we add here the proof that \mathcal{S}_f is a category:

- * identity: $f \bullet id_\Sigma = f = id_{\Sigma'} \bullet f$:

$$(f \bullet id_\Sigma)_n^\# = \check{f} \upharpoonright_n \circ (id_\Sigma)_n^\# = \check{f} \upharpoonright_n \circ (j_\Sigma)_n = (f^\#)_n;$$

$$(id_{\Sigma'} \bullet f)_n^\# = id_{\Sigma'} \upharpoonright_n \circ (f^\#)_n = id_{F(\Sigma')[n]} \circ (f^\#)_n = (f^\#)_n.$$

- * associativity: $(f'' \bullet f') \bullet f = f'' \bullet (f' \bullet f)$:

$$((f'' \bullet f') \bullet f)_n^\# = (f'' \bullet \check{f}') \upharpoonright_n \circ (f^\#)_n = (\check{f}'' \circ \check{f}') \upharpoonright_n \circ (\check{f} \upharpoonright_n \circ (j_\Sigma)_n) =$$

$$\check{f}'' \upharpoonright_n \circ (\check{f}' \upharpoonright_n \circ \check{f} \upharpoonright_n) \circ (j_\Sigma)_n = \check{f}'' \upharpoonright_n \circ (\check{f}' \circ \check{f}) \upharpoonright_n \circ (j_\Sigma)_n = \check{f}'' \upharpoonright_n \circ (f' \bullet f) \upharpoonright_n \circ (j_\Sigma)_n =$$

$$\check{f}'' \upharpoonright_n \circ ((f' \bullet f)^\#)_n = (f'' \bullet (f' \bullet f))_n^\#.$$

The notion of extension of \mathcal{S}_f -morphism to formula algebras shares many properties with notion of extension of \mathcal{S}_s -morphism to formula algebras, however:

(0) If $f \in \mathcal{S}_f(\Sigma, \Sigma')$, then are equivalent:

- * $compl(\check{f}(\theta)) \geq compl(\theta)$, any $\theta \in F(\Sigma)$;
- * $f(c_1) \neq x_0$, all $c_1 \in \Sigma_1$.

Now we add some information easily established:

Definition 1.1 A \mathcal{S}_f -morphism $f : \Sigma \longrightarrow \Sigma'$ is regular if $compl(\check{f}(\theta)) \geq compl(\theta)$, any $\theta \in F(\Sigma)$. □

Proposition 1.2 (a) If $f \in \mathcal{S}_f(\Sigma, \Sigma')$, then: f is regular iff $f(c_1) \neq x_0$, all $c_1 \in \Sigma_1$.

(b) The "empty" signature is the unique initial object of \mathcal{S}_f (as in \mathcal{S}_s).

(c) A (non full) subcategory of \mathcal{S}_f with the same objects has a strict initial object iff all the morphisms are regular.

(d) The mapping $f \in \mathcal{S}_s(\Sigma, \Sigma') \mapsto (j_{\Sigma'} \circ f)^\flat \in \mathcal{S}_f(\Sigma, \Sigma')$ is a (natural) bijection $\mathcal{S}_s(\Sigma, \Sigma') \xrightarrow{\cong} \{h \in \mathcal{S}_f(\Sigma, \Sigma') : compl(\check{h}(\theta)) = compl(\theta), \text{ for any } \theta \in F(\Sigma)\}$.

Proof. The non trivial implication in item (a) follows from induction on complexity of formulas. □

Proposition 1.3 (a) \mathcal{S}_f has weak terminal objects. More precisely, a signature Σ is an weak terminal object iff $\Sigma'_0 \neq \emptyset$ and exists $k \geq 2$ such that $\Sigma'_k \neq \emptyset$.

(b) \mathcal{S}_f does not have terminal object. (Example: let Σ' be a weak terminal object and take a signature Σ with only one connective and it is binary: as $F(\Sigma')[2]$ is infinity, there are many \mathcal{S}_f -morphisms from Σ into Σ' .) \square

Remark 1.4

(a) It is easy to see that \mathcal{S}_f has weak products: a weak product of a (small) family of signatures can be given by taking the product signature in the *strict category* \mathcal{S}_s and the corresponding \mathcal{S}_s -projections, transformed into \mathcal{S}_f -morphisms (see the next subsection).

(b) As \mathcal{S}_f has initial object, any family of parallel arrows has an weak equalizer.

(c) \mathcal{S}_f has weak terminal object but do not have terminal object: a necessary and sufficient condition to Σ be a weak terminal is $\Sigma_0 \neq \emptyset, \text{exists } k \geq 2 \text{ Sigma}_k \neq \emptyset$ (d) \mathcal{S}_f only has "trivial" (i.e. in \mathcal{S}_s) coequalizers, idempotents, isomorphisms, \square

1.2 The fundamental adjunction

Proposition 1.5 Connecting categories of signatures:

(a) We have the (faithful) functors:

$$(+): \mathcal{S}_s \longrightarrow \mathcal{S}_f \quad : \quad (\Sigma \xrightarrow{f} \Sigma') \mapsto (\Sigma \xrightarrow{(j_{\Sigma'} \circ f)^b} \Sigma');$$

$$(-): \mathcal{S}_f \longrightarrow \mathcal{S}_s \quad : \quad (\Sigma \xrightarrow{h} \Sigma') \mapsto ((F(\Sigma)[n])_{n \in \mathbb{N}} \xrightarrow{(\hat{h}|_n)_{n \in \mathbb{N}}} (F(\Sigma')[n])_{n \in \mathbb{N}}).$$

(b) For each $f \in \mathcal{S}_s(\Sigma, \Sigma')$, we have $(f^+) = \hat{f} \in \text{Set}(F(\Sigma), F(\Sigma'))$.

(c) We have the natural transformations:

$$\eta: Id_{\mathcal{S}_s} \longrightarrow (-) \circ (+) \quad : \quad (\eta_{\Sigma})_n := (j_{\Sigma})_n \quad \varepsilon: (+) \circ (-) \longrightarrow Id_{\mathcal{S}_f} \quad : \quad (\varepsilon_{\Sigma})_n^{\sharp} := id_{F(\Sigma)[n]}$$

and we write $\mu = (-)\varepsilon(+)$. \square

Theorem 1.6 The (faithful) functor $(+)$ is a left adjoint of the (faithful) functor $(-)$: η and ε are, respectively, the unit and the counit of the adjunction. \square

Corollary 1.7 (a) The functor $(+)$ preserves colimits and the functor $(-)$ preserves limits. \square

Corollary 1.8 The category \mathcal{S}_f has colimits for any (small) diagram "in \mathcal{S}_s ", i.e., given \mathcal{I} a small category and a diagram $D: \mathcal{I} \longrightarrow \mathcal{S}_s$, the category \mathcal{S}_f has a colimit for the diagram $(+) \circ D: \mathcal{I} \longrightarrow \mathcal{S}_f$. In particular, \mathcal{S}_f has all (small) coproducts and all (small) pushouts "based in \mathcal{S}_s ". \square

Proposition 1.9 Let $h \in \mathcal{S}_f(\Sigma, \Sigma')$:

(a) If h^- is a \mathcal{S}_s -epimorphism, then f is \mathcal{S}_f -epimorphism.

(b) h is a \mathcal{S}_f -monomorphism if and only if h^- is a \mathcal{S}_s -monomorphism.

(c) If h is an \mathcal{S}_f -isomorphism, then " $h \in \mathcal{S}_s$ ", i.e. there is a (unique) \mathcal{S}_s -(iso)morphism f such that $h = f^-$; in particular, h is regular.

(d) If h is a \mathcal{S}_f -section, then h is regular and if $g \bullet h = id$ for some \mathcal{S}_f -morphism g that is regular over the "image signature of h " (i.e. the signature whose connectives effectively occur in the image of some h_n , $n \in \mathbb{N}$), then " $h \in \mathcal{S}_s$ ". \square

1.3 The monad and its properties

We have a (endo)functor $T: \mathcal{S}_s \longrightarrow \mathcal{S}_s \quad (\Sigma \xrightarrow{f} \Sigma') \xrightarrow{T} ((F(\Sigma)[n])_{n \in \mathbb{N}} \xrightarrow{(\hat{f}|_n)_{n \in \mathbb{N}}} (F(\Sigma')[n])_{n \in \mathbb{N}})$. Clearly, $T = (-) \circ (+)$ and it is a faithful functor. Let $\mathcal{T} = (T, \eta, \mu)$ be the monad (or triple) associated

to the adjunction $(\eta, \varepsilon) : \mathcal{S}_s \xrightleftharpoons[(-)]{(+)} \mathcal{S}_f$.

Proposition 1.10 *The functor T reflects isomorphisms (respectively: monomorphisms, epimorphisms).*

Proof. First remark that, for each signature Σ and $n \in \mathbb{N}$, $(\eta_\Sigma)_n : \Sigma_n \rightarrow F(\Sigma)[n]$ establish a bijection between Σ_n and $\{\theta \in F(\Sigma)_n : \text{compl}(\theta) = 1\}$. Now let $f : \Sigma \rightarrow \Sigma'$ a \mathcal{S}_s -morphism such that $T(f)$ is a \mathcal{S}_s -isomorphism (respectively: a \mathcal{S}_s -monomorphism, a \mathcal{S}_s -epimorphism). Then, for each $n \in \mathbb{N}$, $\hat{f}|_n : F(\Sigma)[n] \rightarrow F(\Sigma')[n]$ is a bijection (respectively: a injection, a surjection) and, as $\text{compl}(\hat{f}(\theta)) = \text{compl}(\theta)$ for each $\theta \in F(\Sigma)[n]$, $\hat{f}|_n$ restricts to a bijection (respectively: a injection, a surjection) between $\{\theta \in F(\Sigma)[n] : \text{compl}(\theta) = 1\}$ and $\{\theta' \in F(\Sigma')[n] : \text{compl}(\theta') = 1\}$. Finally, as $\eta_{\Sigma'} \circ f = T(f) \circ \eta_\Sigma$, we conclude that $f_n : \Sigma_n \rightarrow \Sigma'_n$ is a bijection (respectively: a injection, a surjection), for each $n \in \mathbb{N}$, as we need. \square

Proposition 1.11 *The functor T preserves directed colimits (i.e., colimits of diagrams over upward directed posets). More explicitly, let (I, \leq) be an upward directed poset and $D : (I, \leq) \rightarrow \mathcal{S}_s : i \mapsto \Sigma_i$ be a diagram in \mathcal{S}_s ; $(\Sigma, (\Sigma_i \xrightarrow{\alpha_i} \Sigma)_{i \in I})$ denotes the colimit of D in \mathcal{S}_s ; $(\Sigma', (T(\Sigma_i) \xrightarrow{\alpha'_i} \Sigma')_{i \in I})$ be the colimit of $T \circ D$ in \mathcal{S}_s ; $(S, (F(\Sigma_i) \xrightarrow{\beta_i} S)_{i \in I})$ denotes the colimit of $(\hat{\cdot}) \circ D$ in the category Set , then:*

- (a) *The canonical function $S \rightarrow F(\Sigma)$, denoted $k : \text{colim}_{i \in I} F(\Sigma_i) \rightarrow F(\text{colim}_{i \in I} \Sigma_i)$, i.e. the unique function such that $k \circ \beta_i = \hat{\alpha}_i$, $i \in I$, is a bijection.*
- (b) *The canonical \mathcal{S}_s -morphism $\text{can} : \text{colim}_{i \in I} T(\Sigma_i) \rightarrow T(\text{colim}_{i \in I} \Sigma_i)$, i.e. the unique \mathcal{S}_s -morphism such that $\text{can} \circ \alpha'_i = T(\alpha_i)$, $i \in I$, is a \mathcal{S}_s -isomorphism. It is given by sequence of bijections $\text{can}_n : \text{colim}_{i \in I} (F(\Sigma_i)[n]) \rightarrow F(\text{colim}_{i \in I} \Sigma_i)[n]$, $n \in \mathbb{N}$, obtained from the "restrictions" of the canonical bijection k just above.*

Proof. (Sketch) For a proof of item (a) we apply a "global reasoning": we consider formula algebras and apply induction on complexity of formulas. For (b): we extract "local" information from the result is (a), i.e., we consider convenient "restrictions" to the subsets $F(\Sigma)[n]$, $n \in \mathbb{N}$. \square

The same technique of proof in the Proposition above gives us the Theorem below:

Theorem 1.12 *Let $\mathcal{T} = (T, \eta, \mu)$ be the monad associated to the adjunction $(\eta, \varepsilon) : \mathcal{S}_s \xrightleftharpoons[(-)]{(+)} \mathcal{S}_f$ (i.e., $\mu = (-)\varepsilon(+)$) is such that $\text{Kleisli}(\mathcal{T}) = \mathcal{S}_f$. Moreover, the functors $(+)$ and $(-)$ are precisely the canonical functors associated to the adjunction of the Kleisli category of a monad. More explicitly: given $(\Sigma \xrightarrow{f} \Sigma' \xrightarrow{f'} \Sigma'')$ in \mathcal{S}_f , then $f' \bullet f = (\mu_{\Sigma''} \circ T(f^\#) \circ f^\#)^\flat$, i.e., we have in \mathcal{S}_s :*

$$(\Sigma \xrightarrow{f^\#} T(\Sigma') \xrightarrow{(\hat{f}'|_n)_{n \in \mathbb{N}}} T(\Sigma'')) = (\Sigma \xrightarrow{f^\#} T(\Sigma') \xrightarrow{T(f^\#)} T \circ T(\Sigma'') \xrightarrow{\mu_{\Sigma''}} T(\Sigma'')). \quad \square$$

2 Categories of Logics

2.1 The lattice of logics above a signature

A logic is an ordered pair $l = (\Sigma, \vdash)$ where Σ is an object of \mathcal{S}_s and \vdash codifies the (Tarskian) "consequence operator" on $F(\Sigma) : \vdash$ is a binary relation, a subset of $\text{Parts}(F(\Sigma)) \times F(\Sigma)$, such that $\text{Cons}(\Gamma) = \{\varphi \in F(\Sigma) : \Gamma \vdash \varphi\}$, for all $\Gamma \subseteq F(\Sigma)$, gives a structural finitary closure operator on $F(\Sigma)$:

(a) *inflationary:* $\Gamma \subseteq \text{Cons}(\Gamma)$;

- (b) *increasing*: $\Gamma_0 \subseteq \Gamma_1 \Rightarrow \text{Cons}(\Gamma_0) \subseteq \text{Cons}(\Gamma_1)$;
- (c) *idempotent*: $\text{Cons}(\text{Cons}(\Gamma)) \subseteq \text{Cons}(\Gamma)$;
- (d) *finitary*: $\text{Cons}(\Gamma) = \bigcup \{ \text{Cons}(\Gamma') : \Gamma' \subseteq_{\text{fin}} \Gamma \}$;
- (e) *structural*: $\tilde{\sigma}(\text{Cons}(\Gamma)) \subseteq \text{Cons}(\tilde{\sigma}(\Gamma))$, for each substitution $\sigma : X \rightarrow F(\Sigma)$.

The set of all consequence relations on a signature Σ , denoted by Cons_Σ , is endowed with the partial order: $\vdash_0 \leq \vdash_1$ iff for each $\Gamma \in \text{Parts}(F(\Sigma))$, $\bar{\Gamma}^0 \subseteq \bar{\Gamma}^1$.

Fact 2.1 *For each signature Σ , the poset $(\text{Cons}_\Sigma, \leq)$ is a complete lattice. It is in fact an algebraic lattice where the compact elements are the “finitely generated logics”, the logics over Σ given by a finite set of axioms and a finite set of (finitary) inference rules.*

Proof.(sketch)

Infs: Consider I a set and $D = \{l^i = (\Sigma, \vdash_i)\}_{i \in I}$ a family of logics over the signature Σ . Now, for each $\Gamma \cup \{\psi\} \subseteq F(\Sigma)$, define that $\Gamma \vdash \psi \Leftrightarrow$ there is $\Gamma' \subseteq_{\text{fin}} \Gamma$ such that $(\forall i \in I)(\Gamma' \vdash_i \varphi)$, then (Σ, \vdash) is a logic and $l = (\Sigma, \vdash)$ is the infimum of the family D in \mathcal{L} , thus $(\text{Cons}_\Sigma, \leq)$ is a *complete lattice*.

Generated consequence relation: As the set of consequence operators (or consequence relations) on a signature Σ is a complete lattice, there exists the logic generated by any function $W : \text{Parts}(F(\Sigma)) \rightarrow \text{Parts}(F(\Sigma))$: it is enough to take the infimum of the family of all consequence relations on Σ that are upper bounds of the “proto-consequence relation” $W \upharpoonright : \text{Parts}_{\text{fin}}(F(\Sigma)) \rightarrow \text{Parts}(F(\Sigma))$ associated with W . A more explicit description is given by the usual notion of “proof” based on hypothesis, axioms and inference rules.

Directed sups: Consider I a set and $D = \{l^i = (\Sigma, \vdash_i)\}_{i \in I}$ an upward directed family of logics over the signature Σ , that is, for each $i, j \in I$ there is a $k \in I$ such that $id_\Sigma \in \mathcal{L}(l^i, l^k)$, $id_\Sigma \in \mathcal{L}(l^j, l^k)$. Now, for each $\Gamma \cup \{\psi\} \subseteq F(\Sigma)$, define that $\Gamma \vdash \psi \Leftrightarrow$ there is $\Gamma' \subseteq_{\text{fin}} \Gamma$ and there is an $i \in I$ such that $\Gamma' \vdash_i \varphi$, then (Σ, \vdash) is a logic and $l = (\Sigma, \vdash)$ is the supremum of the family D in \mathcal{L} .

Sups: As usual, the supremum of a family of logics can be obtained taking the infimum of the set of upper bounds of that family of logics. A more objective characterization of suprema can be given but we postpone that because this can be easily described by more general results below (see Proposition 2.5).

Compact consequence relations: A consequence relation \vdash' over Σ is compact if for each set I , each $D = \{l^i = (\Sigma, \vdash_i)\}_{i \in I}$ a upward directed family of logics over the signature Σ , if $\vdash' \leq \bigvee_{i \in I} \vdash_i$ then there is an $i \in I$ such that $\vdash' \leq \vdash_i$. It follows easily that this condition is equivalent to the “stronger” condition: for each set J , each $D = \{l^j = (\Sigma, \vdash_j)\}_{j \in J}$ a family of logics over the signature Σ , if $\vdash' \leq \bigvee_{j \in J} \vdash_j$ then there is a finite subset $J' \subseteq J$ such that $\vdash' \leq \bigvee_{j \in J'} \vdash_j$.² A consequence relation on Σ is compact if and only if it is a finitely generated consequence relation on Σ . Any consequence relation on Σ is the directed supremum of its compact (sub)consequence relations on Σ . \square

2.2 Known facts about categories of logics

The category \mathcal{L}_s is the category of propositional logics and *strict* translations as morphisms. This is a category “built above” the category \mathcal{S}_s , that is, there is an obvious forgetful functor $U_s : \mathcal{L}_s \rightarrow \mathcal{S}_s$. The categorial properties of \mathcal{L}_s are detailed in [AFLM3].

The objects of \mathcal{L}_s are logics $l = (\Sigma, \vdash)$ as described in subsection 2.1.

²Just observe that any sup of a family coincides with a sup of a directed family: for each set J take $I = P_{\text{fin}}(J)$ then, for each $J' \subseteq_{\text{fin}} J$, define $\vdash_{J'} = \bigvee_{j \in J'} \vdash_j \dots$

If $l = (\Sigma, \vdash), l' = (\Sigma', \vdash')$ are logics then a *strict translation morphism* $f : l \rightarrow l'$ in \mathcal{L}_s is a *strict signature morphism* $f : \Sigma \rightarrow \Sigma'$ in \mathcal{S}_s such that “preserves the consequence relation”, that is, for all $\Gamma \cup \{\psi\} \subseteq F(\Sigma)$, if $\Gamma \vdash \psi$ then $\widehat{f}[\Gamma] \vdash' \widehat{f}(\psi)$. Composition and identities are similar to \mathcal{S}_s .

\mathcal{L}_s has natural notions of direct and inverse image logics under a \mathcal{S}_s -morphism and they have good properties:

2.2 Direct image and inverse image:

Let $f : \Sigma \rightarrow \Sigma'$ be a \mathcal{S}_s -morphism:

Inverse image: if $l' = (\Sigma', \vdash') \in \text{Obj}(\mathcal{L})$ then for all $\Gamma \cup \{\psi\} \subseteq F(\Sigma)$ define $\Gamma \vdash_{f^*(\vdash')} \psi$ iff $\widehat{f}[\Gamma] \vdash' \widehat{f}(\psi)$;

Direct image: if $l = (\Sigma, \vdash) \in \text{Obj}(\mathcal{L})$ then for all $\Gamma' \cup \{\psi'\} \subseteq F(\Sigma')$ define $\Gamma' \vdash_{f_*(\vdash)} \psi'$ iff there is a finite sequence of Σ' -formulas $(\phi'_0, \dots, \phi'_t)$ such that:

- $\phi'_t = \psi'$;
- for all $p \leq t$ at least one of the alternatives below occurs:
 - * “ ϕ'_p is a hypothesis”: $\phi'_p \in \Gamma'$;
 - * “ ϕ'_p is an instance of an l -axiom”: there is a $\theta_p \in F(\Sigma)$ such that $\vdash \theta_p$ and there is a substitution $\sigma' : X \rightarrow F(\Sigma')$ such that $\sigma'(\widehat{f}(\theta_p)) = \phi'_p$;
 - * “ ϕ'_p is a direct consequence of an instance of l -inference rule applied over previous members in the sequence”: there is a $\Delta_p \cup \{\theta_p\} \subseteq_{fin} F(\Sigma)$ such that $\Delta_p \vdash \theta_p$ and there is a substitution $\sigma' : X \rightarrow F(\Sigma')$ such that $\sigma'(\widehat{f}(\theta_p)) = \phi'_p$ and $\sigma'[\widehat{f}[\Delta_p]] \subseteq \{\phi'_0, \dots, \phi'_{p-1}\}$. \square

Fact 2.3 Let $f : \Sigma \rightarrow \Sigma'$ be a \mathcal{S}_s -morphism and let $l = (\Sigma, \vdash), l' = (\Sigma', \vdash')$ be logics $l, l' \in \text{Obj}(\mathcal{L}_s)$. Then

- (i) * and (i) $_{\star}$ hold, and (ii) * , (ii) $_m$ and (ii) $_{\star}$ are equivalent:
- (i) * if $l' = (\Sigma', \vdash') \in \text{Obj}(\mathcal{L}_s)$ then $f^*(l') = (\Sigma, \vdash_{f^*(\vdash')}) \in \text{Obj}(\mathcal{L}_s)$;
- (i) $_{\star}$ if $l = (\Sigma, \vdash) \in \text{Obj}(\mathcal{L}_s)$ then $f_*(l) = (\Sigma', \vdash_{f_*(\vdash)}) \in \text{Obj}(\mathcal{L}_s)$.
- (ii) * $\vdash \leq f^*(\vdash')$;
- (ii) $_m$ $f : (\Sigma, \vdash) \rightarrow (\Sigma', \vdash')$ is \mathcal{L}_s -morphism;
- (ii) $_{\star}$ $f_*(\vdash) \leq' \vdash'$. \square

Remark 2.4 It follows easily from the facts above that the forgetful functor $U_s : \mathcal{L}_s \rightarrow \mathcal{S}_s : ((\Sigma, \vdash) \xrightarrow{f} (\Sigma', \vdash')) \mapsto (\Sigma \xrightarrow{f} \Sigma')$ has left and right adjoint functors: the left adjoint $\perp_s : \mathcal{S}_s \rightarrow \mathcal{L}_s$ and the right adjoint $\top_s : \mathcal{S}_s \rightarrow \mathcal{L}_s$ take a signature Σ to, respectively, $\perp_s(\Sigma) = (\Sigma, \vdash_{min})$ (the first element of Cons_{Σ}) and $\top_s(\Sigma) = (\Sigma, \vdash_{max})$ (the last element of Cons_{Σ}). Moreover, $U_s \circ \perp_s = \text{Id}_{\mathcal{S}_s} = U_s \circ \top_s$ and U_s preserves all limits and colimits that exists in \mathcal{S}_s . \square

Fact 2.5 The category \mathcal{L}_s is complete and cocomplete and the forgetful functor $U_s : \mathcal{L}_s \rightarrow \mathcal{S}_s$ “lifts” all small limits and colimits. \square

2.6 We describe explicitly limits, directed colimits and (general) colimits in \mathcal{L}_s :

Limits: Let \mathcal{I} be a small category and $D : \mathcal{I} \rightarrow \mathcal{L}, ((\Sigma^i, \vdash_i) \xrightarrow{f^h} (\Sigma^j, \vdash_j))_{(h:i \rightarrow j) \in \mathcal{I}}$ a diagram, and take $(\Sigma, (\pi^i)_{i \in \text{Obj}(\mathcal{I})})$ the limit of the underlying diagram $(\mathcal{I} \xrightarrow{D} \mathcal{S} \xrightarrow{U} \mathcal{L})$. For all $\Gamma \cup \{\psi\} \subseteq F(\Sigma)$, define that $\Gamma \vdash \psi \Leftrightarrow$ there is $\Gamma^- \subseteq_{fin} \Gamma$ such that for all $i \in \text{Obj}(\mathcal{I})$ $\widehat{\pi}^i[\Gamma^-] \vdash_i \widehat{\pi}^i(\psi)^3$, then $l = (\Sigma, \vdash)$ is a logic and $(l, (\pi^i)_{i \in \text{Obj}(\mathcal{I})})$ is the limit of D in \mathcal{L} .

Directed colimits: Let (I, \leq) be a directed ordered set and $D : (I, \leq) \rightarrow \mathcal{L}, ((\Sigma^i, \vdash_i) \xrightarrow{f^{ij}} (\Sigma^j, \vdash_j))_{(i \leq j) \in I}$ be a diagram. Take $(\Sigma, (\gamma^i)_{i \in I})$ the colimit of the underlying diagram

³This definition also works for the terminal logic $l = (\Sigma, \vdash)$ where Σ is the terminal signature ($\text{card}(\Sigma_n) = 1, \forall n \in \omega$), and for all $\Gamma \cup \{\psi\} \subseteq F(\Sigma)$, $\Gamma \vdash \psi$.

$(\mathcal{I} \xrightarrow{D} \mathcal{S} \xrightarrow{U} \mathcal{L})$. Now, for all $\Gamma \cup \{\psi\} \subseteq F(\Sigma)$, define that $\Gamma \vdash \psi \Leftrightarrow$ there is $\Gamma^- \subseteq_{fin} \Gamma$ and there is an $i \in I$ such that $\Gamma^- \cup \{\psi\} \subseteq \hat{\gamma}^i[F(\Sigma^i)]$ and there is $\Gamma^{-i} \cup \{\psi^i\} \subseteq_{fin} F(\Sigma^i)$ such that $\hat{\gamma}^i[\Gamma^{-i}] = \Gamma^-$, $\hat{\gamma}^i(\psi^i) = \psi$ and $\Gamma^{-i} \vdash_i \psi^i$. Then $l = (\Sigma, \vdash)$ is a logic and $(l, (\gamma^i)_{i \in I})$ is the colimit of D in \mathcal{L} .

Colimits: Let \mathcal{I} be a small category and $D : \mathcal{I} \rightarrow \mathcal{L}$, $((\Sigma^i, \vdash_i) \xrightarrow{f^h} (\Sigma^j, \vdash_j))_{(h:i \rightarrow j) \in \mathcal{I}}$ be a diagram, and take $(\Sigma, (\gamma^i)_{i \in Obj(\mathcal{I})})$ the colimit of the underlying diagram $(\mathcal{I} \xrightarrow{D} \mathcal{S} \xrightarrow{U} \mathcal{L})$. Now, for all $\Gamma \cup \{\psi\} \subseteq F(\Sigma)$, define that $\Gamma \vdash \psi \Leftrightarrow$ there is a finite sequence of Σ -formulas (ϕ_0, \dots, ϕ_t) , where $\phi_t = \psi$ and for all $p \leq t$ one of these alternative occurs:

- “ ϕ_p is an hypothesis”: $\phi_p \in \Gamma$;
 - “ ϕ_p is an axiom”: there are $i \in Obj(\mathcal{I})$, $\theta^i \in F(\Sigma^i)$, $\sigma : X \rightarrow F(\Sigma)$ such that $\vdash_i \theta^i$ and $\phi_p = \tilde{\sigma}(\hat{\gamma}^i(\theta^i))$;
 - “ ϕ_p is a consequence of a inference rule”: there are $i \in Obj(\mathcal{I})$, $\Delta^i \cup \{\theta^i\} \subseteq_{fin} F(\Sigma^i)$, $\sigma : X \rightarrow F(\Sigma)$ such that $\Delta^i \vdash_i \theta^i$ and $\tilde{\sigma}[\hat{\gamma}^i[\Delta^i]] \subseteq \{\phi_0, \dots, \phi_{p-1}\}$, $\phi_p = \tilde{\sigma}(\hat{\gamma}^i(\theta^i))$;
- Then $l = (\Sigma, \vdash)$ is a logic and $(l, (\gamma^i)_{i \in Obj(\mathcal{I})})$ is the colimit of D in \mathcal{L} . \square

Fact 2.7 The category \mathcal{L}_s is a finitely locally presentable category, i.e., \mathcal{L}_s is a finitely accessible category that is cocomplete and/or complete ([AR]). The finitely presentable objects in \mathcal{L}_s are precisely the logics $l = (\Sigma, \vdash)$ with Σ finitely presentable in \mathcal{S}_s and \vdash is a compact consequence relation in $Cons_\Sigma$. \square

Remark 2.8 In the sequence of works, [AFLM1], [AFLM2], [AFLM3] is proven that the category \mathcal{A}_s of Blok-Pigozzi algebraizable logics ([BP]) and \mathcal{L}_s -morphisms that induces algebraizing pairs preserving functions on the formula algebras is a relatively complete ω -accessible category ([AR]). \square

Remark 2.9 The fundamental defect of \mathcal{L}_s is that the presentations of classical logic, for instance in the signatures $\Sigma = (\neg, \rightarrow)$ and $\Sigma' = (\neg', \vee')$, are not \mathcal{L}_s -isomorphic: this deficiency is inherited from \mathcal{S}_s , because the \mathcal{S}_s -morphisms are too strict. \square

2.3 New results on categories of signatures and of logics

The category \mathcal{L}_f is the category of propositional logics and *flexible* translations as morphisms. This is a category “built above” the category \mathcal{S}_f , that is, there is an obvious forgetful functor $U_f : \mathcal{L}_f \rightarrow \mathcal{S}_f$. The categories \mathcal{S}_f and \mathcal{L}_f are considered in the literature ([BC], [BCC1], [BCC2], [CG], [FC]), but with a different emphasis: Here we provide a more systematic analysis of category \mathcal{L}_f and its relation with \mathcal{S}_f and \mathcal{L}_s .

The objects of \mathcal{L}_f are logics $l = (\Sigma, \vdash)$, as described in subsection 2.1.

If $l = (\Sigma, \vdash), l' = (\Sigma', \vdash')$ are logics then a *flexible translation morphism* $h : l \rightarrow l'$ in \mathcal{L}_f is a *flexible* signature morphism $f : \Sigma \rightarrow \Sigma'$ in \mathcal{S}_f such that “preserves the consequence relation”, that is, for all $\Gamma \cup \{\psi\} \subseteq F(\Sigma)$, if $\Gamma \vdash \psi$ then $\bigvee_h [\Gamma] \vdash_{f^*}^{\bigvee_h} (\psi)$. Composition and identities are similar to \mathcal{S}_f .

As in \mathcal{L}_s , \mathcal{L}_f has natural notions of direct and inverse image logics under a \mathcal{S}_f -morphism (just replace \hat{f} by \check{f}) and they have good properties. For instance:

Fact 2.10 Let $f : \Sigma \rightarrow \Sigma'$ be a \mathcal{S}_f -morphism and let $l = (\Sigma, \vdash), l' = (\Sigma', \vdash')$ be logics $l, l' \in Obj(\mathcal{L}_f)$. Then (i) * and (i) $_*$ hold, and (ii) * , (ii) $_m$ and (ii) $_*$ are equivalent:

(i) * if $l' = (\Sigma', \vdash') \in Obj(\mathcal{L}_f)$ then $f^*(l') = (\Sigma, \vdash_{f^*(\vdash')}) \in Obj(\mathcal{L}_f)$;

(i) $_*$ if $l = (\Sigma, \vdash) \in Obj(\mathcal{L}_f)$ then $f_*(l) = (\Sigma', \vdash_{f_*(\vdash)}) \in Obj(\mathcal{L}_f)$.

$$(ii)^* \vdash \leq f^*(\vdash');$$

$$(ii)_m f : (\Sigma, \vdash) \longrightarrow (\Sigma', \vdash') \text{ is } \mathcal{L}_f\text{-morphism};$$

$$(ii)_* f_*(\vdash) \leq' \vdash'. \quad \square$$

Remark 2.11 It follows easily from the facts above that the forgetful functor $U_f : \mathcal{L}_f \longrightarrow \mathcal{S}_f : ((\Sigma, \vdash) \xrightarrow{h} (\Sigma', \vdash')) \mapsto (\Sigma \xrightarrow{h} \Sigma')$ has left and right adjoint functors: the left adjoint $\perp_f : \mathcal{S}_s \longrightarrow \mathcal{L}_s$ and the right adjoint $\top_f : \mathcal{S}_s \longrightarrow \mathcal{L}_s$ take a signature Σ to, respectively, $\perp_f(\Sigma) = (\Sigma, \vdash_{min})$ (the first element of $Cons_\Sigma$) and $\top_f(\Sigma) = (\Sigma, \vdash_{max})$ (the last element of $Cons_\Sigma$). Moreover, $U_f \circ \perp_f = Id_{\mathcal{S}_f} = U_f \circ \top_f$ and U_f preserves all limits and colimits that exists in \mathcal{S}_f . \square

Remark 2.12 It is known that \mathcal{L}_f has weak products, coproducts and some pushouts, and in the Remark above we see that U_f preserves limits and colimits. As U_f also "lift" limits and colimits – the constructions in \mathcal{L}_f are analogous to in \mathcal{L}_s , just replace \hat{f} by \check{f} – then given a small category \mathcal{I} , \mathcal{L}_f is \mathcal{I} -complete (respectively, \mathcal{I} -cocomplete) *if and only if* \mathcal{S}_f is \mathcal{I} -complete (respectively, \mathcal{I} -cocomplete). Thus the Corollary 1.8 entails that \mathcal{L}_f has colimits for any (small) diagram "in \mathcal{L}_s " (i.e., obtained via $(+) : \mathcal{S}_s \longrightarrow \mathcal{S}_f$), in particular, it has all unconstrained fibrings (= coproducts) and the constrained fibrings (= pushouts) "based in \mathcal{L}_s ". \square

Remark 2.13 The fact of the formula algebra functions induced by \mathcal{S}_f -morphisms "increase complexity" (see Proposition 1.2.(a) for the precise statement) impose many limitations on \mathcal{L}_f . For instance:

- (a) In [CG] is shown that \mathcal{L}_f solves the identity problem for the presentations of classical logic in terms of the (weaker) concept of *equipollence of logics*⁴. But \mathcal{L}_f does not solve problem of identity for the presentations of classical logic in terms of \mathcal{L}_f -isomorphisms.
- (b) \mathcal{L}_f has weak terminal object but does not have terminal object; analogous statements holds in general for \mathcal{L}_f concerning (weak) products. \square

The results below, together with 2.11, 2.12, constitute a strong evidence that the all defects in \mathcal{L}_f are inherited from \mathcal{S}_f .

Theorem 2.14 The signature adjunction $\mathcal{S}_s \xrightleftharpoons[(-)_S]{(+)_S} \mathcal{S}_f$ "lifts", via the forgetful functors U_s and U_f , to a

logic adjunction $\mathcal{L}_s \xrightleftharpoons[(-)_L]{(+)_L} \mathcal{L}_f$, i.e.:

- $U_f \circ (+)_L = (+)_S \circ U_s$;
- $U_s \circ (-)_L = (-)_S \circ U_f$;
- $U_s \eta_L = \eta_S U_s$;
- $U_f \varepsilon_L = \varepsilon_S U_f$.

Moreover, the following relations hold:

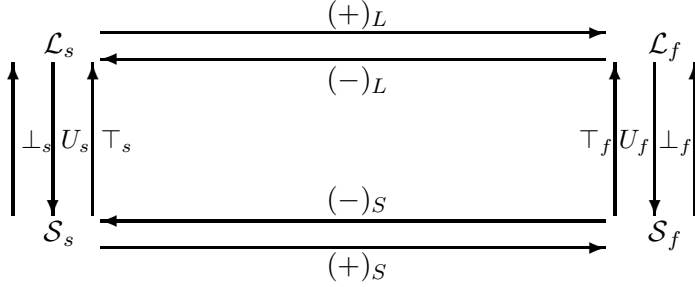
- $(+)_L \circ \perp_s = \perp_f \circ (+)_S$;
- $(+)_L \circ \top_s = \top_f \circ (+)_S$;
- $(-)_L \circ \perp_f = \perp_s \circ (-)_S$;
- $(-)_L \circ \top_f \leq \top_s \circ (-)_S$.

Proof. We provide only (in this moment), the definitions of the (faithful) functors:

$$(+)_L : \mathcal{L}_s \longrightarrow \mathcal{L}_f :$$

⁴We thank professor Marcelo Coniglio for that reference.

$$\begin{aligned}
((\Sigma, \vdash) \xrightarrow{f} (\Sigma', \vdash')) &\mapsto ((\Sigma, \vdash) \xrightarrow{(j_{\Sigma'}^{circf})^\flat} (\Sigma', \vdash')); \\
(-)_L : \mathcal{L}_f &\longrightarrow \mathcal{L}_s : \\
((\Sigma, \vdash) \xrightarrow{h} (\Sigma', \vdash')) &\mapsto (((F(\Sigma)[n])_{n \in \mathbb{N}}, (j_\Sigma)_*(\vdash)) \xrightarrow{(\tilde{h}|_n)_{n \in \mathbb{N}}} ((F(\Sigma')[n])_{n \in \mathbb{N}}, (j_{\Sigma'})_*(\vdash'))). \quad \square
\end{aligned}$$



Theorem 2.15 *The signature monad $\mathcal{T}_S = (T_S, \eta_S, \mu_S)$ associated to the signature adjunction $(\eta_S, \varepsilon_S) : \mathcal{S}_s \xrightleftharpoons[(-)_S]{(+)_S} \mathcal{S}_f$ (i.e., $\mu_S = (-)_S \varepsilon_S (+)_S$) "lifts" to a logic monad $\mathcal{T}_L = (T_L, \eta_L, \mu_L)$ associated to the signature adjunction $(\eta_L, \varepsilon_L) : \mathcal{L}_s \xrightleftharpoons[(-)_L]{(+)_L} \mathcal{L}_f$ (i.e., $\mu_L = (-)_L \varepsilon_L (+)_L$) and is such that $\text{Kleisli}(\mathcal{T}_L) = \mathcal{L}_f$. Moreover, the functors $(+)_L$ and $(-)_L$ are precisely the canonical functors associated to the adjunction of the Kleisli category of a monad.* \square

2.4 The appropriate categories of logics

In 2.9 we saw that the fundamental defect of logical category \mathcal{L}_s is due to the strictness of \mathcal{S}_s -morphisms and, analogously, in the previous subsection we saw that the deficiencies of the logical category \mathcal{L}_f are inherited from the signature category \mathcal{S}_f . Here we introduce new categories of logics, still with logics as objects, one of them satisfies *simultaneously* all the four natural requirements described in the Introduction.

2.16 We will write \mathcal{Q}_f for the **q**uotient category of \mathcal{L}_f by the relation of interdemonstrability: the objects of \mathcal{Q}_f are the logics $l = (\Sigma, \vdash)$ and $\mathcal{Q}_f((\Sigma, \vdash), (\Sigma', \vdash')) := \{[f] : f \in \mathcal{L}_f((\Sigma, \vdash), (\Sigma', \vdash'))\}$, where $[f] := \{g \in \mathcal{L}_f((\Sigma, \vdash), (\Sigma', \vdash')) : f \sim g\}$ and $f \sim g$ iff $(f \dashv \vdash) = (g \dashv \vdash) : F(\Sigma) / \dashv \vdash \longrightarrow F(\Sigma') / \dashv \vdash$. Clearly, the relation \sim is a congruence relation in the category \mathcal{L}_f and we can take $\mathcal{Q}_f := \mathcal{L}_f / \sim$. \square

Fact 2.17 (a) \mathcal{Q}_f has terminal object, coequalizers, weak products and weak coproducts.
(b) The problem of identity for the presentations of classical logic (without logical constants) is solved in terms of \mathcal{Q}_f -isomorphisms.
(c) Any presentation of classical logic l are \mathcal{Q}_f -strongly rigid, i.e., if $f : l \rightarrow l$ is a \mathcal{L}_f -morphism, then $[f] = [id] \in \mathcal{Q}_f(l, l)$.
(d) By Proposition 4.3 in [CG], the logics l, l' are equipollent iff they are \mathcal{Q}_f -isomorphic. \square

2.18 A logic (Σ, \vdash) is *congruential* if, for each $c_n \in \Sigma_n$ and each $\{(\varphi_0, \psi_0), \dots, (\varphi_{n-1}, \psi_{n-1})\}$ such that $\varphi_0 \dashv \vdash \psi_0, \dots, \varphi_{n-1} \dashv \vdash \psi_{n-1}$, then $c_n(\varphi_0, \dots, \varphi_{n-1}) \dashv \vdash c_n(\psi_0, \dots, \psi_{n-1})$. It follows that if $\vartheta_0, \vartheta_1 \in F(\Sigma)$ are such that $\text{var}(\vartheta_0) = \text{var}(\vartheta_1) = \{x_{i_0}, \dots, x_{i_{n-1}}\}$ and $\vartheta_0 \dashv \vdash \vartheta_1$ then $\vartheta_0[\vec{x} \mid \vec{\varphi}] \dashv \vdash \vartheta_1[\vec{x} \mid \vec{\psi}]$. Clearly, the presentations of classical logic are congruential logics. \square

2.19 Denote \mathcal{L}_f^c the full subcategory of \mathcal{L}_f whose objects are the congruential logics. \square

Proposition 2.20 \mathcal{L}_f^c is a reflective subcategory of \mathcal{L}_f , i.e. $i : \mathcal{L}_f^c \hookrightarrow \mathcal{L}_f$ has a left adjoint $c : \mathcal{L}_f \rightarrow \mathcal{L}_f^c$, moreover, the underlying signatures of the logics l and $c(l)$ coincide.

2.21 Analogously to 2.8, denote \mathcal{A}_f , the category of Blok-Pigozzi algebraizable logics and \mathcal{L}_f -morphisms that preserves algebraizable pairs (well defined). Let $Lind(\mathcal{A}_f) \subseteq \mathcal{A}_f$, the full subcategory of Lindenbaum algebraizable logics, i.e. an algebraizable logic $l = (\Sigma, \vdash)$ is *Lindenbaum algebraizable* iff for each $\varphi, \psi \in F(\Sigma)$, $\varphi \Vdash \psi \Leftrightarrow \vdash \varphi \Delta \psi$ (well defined). □

Fact 2.22 (a) $Lind(\mathcal{A}_f) \subseteq \mathcal{L}_f^c$.

(b) $Lind(\mathcal{A}_f) \hookrightarrow \mathcal{A}_f$ is a reflective subcategory. □

2.23 It follows from the Proposition above that \mathcal{L}_f^c has coproducts: it is the "congruential closure" of the coproduct in \mathcal{L}_f of a discrete diagram in \mathcal{L}_f^c . □

2.24 Let \mathcal{Q}_f^c denote the full subcategory of \mathcal{Q}_f whose objects are the congruential logics. Remark that \mathcal{Q}_f^c coincide with $Q(\mathcal{L}_f^c)$, the quotient of the (sub)category \mathcal{L}_f^c . □

We propose that \mathcal{Q}_f^c is a convenient category to perform combinations of logics. The rest of the subsection is devoted to justify this claim.

Fact 2.25 If $i : \mathcal{Q}_f^c \hookrightarrow \mathcal{Q}_f$ is the inclusion functor, then i has a left adjoint $c : \mathcal{Q}_f \rightarrow \mathcal{Q}_f^c$. □

$$\begin{array}{ccc}
 \mathcal{L}_f & \begin{array}{c} \xrightarrow{c} \\ \xleftarrow{i} \end{array} & \mathcal{L}_f^c \\
 \downarrow q_f & & \downarrow q_f^c \\
 \mathcal{Q}_f & \begin{array}{c} \xrightarrow{i} \\ \xrightarrow{c} \end{array} & \mathcal{Q}_f^c
 \end{array}$$

Remark 2.26 (a) The proposition above, gives us a natural interpretation of the "ubiquity" of the logical systems considered.

(b) As in \mathcal{Q}_f , the problem of identity for the presentations of classical logic (without logical constants) is solved in terms of \mathcal{Q}_f^c -isomorphisms.

(c) *Problem:* To describe explicitly congruential closures of not well-behaved logics as paraconsistent logics. □

Proposition 2.27 Given $f \in \mathcal{L}_f^c(l, l')$, then $[f]$ is a \mathcal{Q}_f^c -isomorphism iff f is an "weak equivalence", i.e., it holds:

- conservative translation: $\forall \Gamma \cup \{\psi\} \subseteq F(\Sigma) \quad \Gamma \vdash \psi \Leftrightarrow \check{f}[\Gamma] \vdash' \check{f}(\psi)$;
- denseness: $\forall n \in \mathbb{N} \quad \forall \vartheta' \in F(\Sigma')[n] \quad \exists \vartheta \in F(\Sigma)[n] \quad \vartheta' \dashv\vdash \check{f}(\vartheta)$.

□

2.28 The *coproducts* in \mathcal{Q}_f^c are obtained taking first a cone coproduct in \mathcal{L}_f : the vertex in \mathcal{Q}_f^c is the congruential closure of the vertex in \mathcal{L}_f and the cocone arrows in \mathcal{Q}_f^c are the classes of equivalence of the cocone arrows in \mathcal{L}_f (the congruential property is decisive in proof of uniqueness). The *coequalizers* in \mathcal{Q}_f^c are obtained taking first a cone coequalizer in \mathcal{Q}_f and then taking the induced cone in \mathcal{Q}_f^c obtained by the reflection functor $c : \mathcal{Q}_f \rightarrow \mathcal{Q}_f^c$. □

Proposition 2.29 \mathcal{Q}_f^c is a cocomplete category. \square

2.30 A congruential logics (Σ, \vdash) is of *finite type* if it has a finite support signature ($\text{card}(\bigcup_{n \in \mathbb{N}} \Sigma_n) < \omega$) and is the congruential closure of a consequence relation over Σ that is generated by substitutions with a finite set of axioms and a finite set of (finitary) inference rules. \square

Fact 2.31 There is only an $2^{2^{\aleph_0}}$ set of classes of \mathcal{Q}_f^c -isomorphism of finite type congruential logics. \square

Proposition 2.32 Any congruential logic is a colimit in \mathcal{Q}_f^c of a directed diagram of congruential logics of finite type. \square

Proposition 2.33 Let (I, \leq) be an upward directed poset and let $D : (I, \leq) \rightarrow \mathcal{Q}_f^c$, $i \rightarrow j \mapsto (\Sigma_i, \vdash_i)$ be a diagram. Then $\text{colim}(D) = ((\Sigma_j, \vdash_j) \xrightarrow{[\alpha_j^+]} (\sqcup_{i \in I} \Sigma_i, \vdash)_{j \in I})$, where $\alpha_j \in \mathcal{S}_s(\Sigma_j, \sqcup_{i \in I} \Sigma_i)$ and for each $\Gamma \cup \{\varphi\} \subseteq F(\sqcup_{i \in I} \Sigma_i)$, $\Gamma \vdash \varphi \Leftrightarrow \exists \Gamma' \subseteq_{\text{fin}} \Gamma, \exists j \geq I' = \{i \in I : \text{for some } n \in \mathbb{N} \text{ some } n\text{-ary connective } (c_n, i) \in \sqcup_{i \in I} \Sigma_i \text{ occurs in } \psi \in \Gamma' \cup \{\varphi\}\} \subseteq_{\text{fin}} I, \Gamma'^{(j)} \vdash_j \varphi^{(j)}$, where for $\psi \in \Gamma' \cup \{\varphi\}$ and each subformula θ of ψ :

- $\theta^{(j)} = x_l$, if $\theta = x_l$;
- $\theta^{(j)} = \check{h}_{ij}((c_n, i))[x_0 | \gamma_0^{(j)}, \dots, x_{n-1} | \gamma_{n-1}^{(j)}]$, if $\theta = (c_n, i)(\gamma_0, \dots, \gamma_{n-1})$. \square

Proposition 2.34 In \mathcal{Q}_f^c , a congruential logic of finite type is finitely presentable. Thus a congruential logic is finitely presentable iff it is a retract of a congruential logic of finite type. \square

Theorem 2.35 \mathcal{Q}_f^c is a finitely locally presentable category, i.e. it is a finitely accessible category complete/cocomplete. \square

A fundamental result in the theory of accessible categories (see [AR]) ensures that an accessible category is complete iff its cocomplete. But we will not provide here explicit description of all limits in \mathcal{Q}_f^c !

2.36 A natural notion of (Lindenbaum) algebrized logic is given by the triples $(\Sigma, \vdash, \Delta / \dashv\vdash)$ where $l = (\Sigma, \vdash)$ is a logic and $\Delta \subseteq_{\text{fin}} F(\Sigma)[2]$ is a set of "equivalence formulas in the Lindenbaum sense" i.e.:

- (a) $\vdash \varphi \Delta \varphi$ ⁵;
- (b) $\varphi \Delta \psi \vdash \psi \Delta \varphi$;
- (c) $\varphi \Delta \psi, \psi \Delta \vartheta \vdash \varphi \Delta \vartheta$;
- (d) $\varphi_0 \Delta \psi_0, \dots, \varphi_{n-1} \Delta \psi_{n-1} \vdash c_n(\varphi_0, \dots, \varphi_{n-1}) \Delta c_n(\psi_0, \dots, \psi_{n-1})$;
- (e) $\varphi \dashv\vdash \psi$ iff $\vdash \varphi \Delta \psi$.

Clearly, the underlying logic of $(l, \Delta / \dashv\vdash)$ is congruential. \square

2.37 The corresponding category \mathcal{A} of algebrized logics has as morphisms $f : (l, \Delta / \dashv\vdash) \rightarrow (l', \Delta' / \dashv\vdash)$ the $\Delta' \vdash^{\vee} f[\Delta]$ (or $\Delta' \dashv\vdash^{\vee} f[\Delta]$). Composition and identities are as in \mathcal{Q}_f^c qdr

We finish the section with the following diagram:

⁵That is, if $\Delta = \{\Delta_u : u < v\}$, then $\vdash \varphi \Delta_u \varphi$, for all $u < v$.

$$\begin{array}{ccccc}
\mathcal{A}_f & \xrightarrow{\text{incl}} & \mathcal{L}_f & \xrightarrow{q} & \mathcal{Q}_f \\
\downarrow L & & \downarrow c & & \downarrow \bar{c} \\
\uparrow j & & \uparrow i & & \uparrow \bar{i} \\
\text{Lind}(\mathcal{A}_f) & \xrightarrow{\text{incl}} & \mathcal{L}_f^c & \xrightarrow{q^c} & \mathcal{Q}_f^c
\end{array}$$

3 Final remarks and future works

3.1 It should be remarked that all the categories of logics considered have the same objects (of combinatorial nature) $l = (\Sigma, \vdash)$ and the morphisms in all that categories area, in some sense, translation morphisms: thus we believe that the new categories introduced $(\mathcal{L}_f^c, \text{Lind}(\mathcal{A}_f), \mathcal{Q}_f, \mathcal{Q}_f^c)$ constitute a natural and simple solution of the deficiencies on \mathcal{L}_s and \mathcal{L}_f , that satisfies the natural requirements (i),..., (iv) in the Introduction. \square

3.2 In the definition of the quotient category \mathcal{Q}_f (respec. \mathcal{Q}_f^c), is considered a congruence on category \mathcal{L}_f (respec. \mathcal{L}_f^c), that is induced by a *pre-order* relation on the category: thus \mathcal{Q}_f (respec. \mathcal{Q}_f^c) has a natural structure of category enriched by the category of posets (and increasing functions) and this should be explored in the future. \square

3.3 Another possible approach to overcome the deficiencies of the categories of logics that are inherited from the defects of the underlying categories of [AFLM2], [AFLM3]): A possible notion candidate for this task is concept of *operad*, as mentioned in the section 5 of [AFLM3]. An operad can be understood as an axiomatization of the behavior of a collection of finitary operations on a set and such that is closed by the formation of the derived operations by composition. A candidate for the category of signatures would be the category of all operads: a category that somehow contains the categories proposed in the two series of papers mentioneds, but that it has good categorial properties (is complete, is cocomplete,...). Moreover the category of logical systems built on this new category of signatures would allow, in principle, the interdefinability of connectives as now the operations may satisfy relations, something that would promote a satisfactory treatment of the problem of identity. \square

3.4 In [MP] is in developement an alternative to overcome the problems by a mathematical device generically called "Morita equivalence of logics", a notion borrowed from Ring and Module Theories: In the representation theory of rings, the category of rings is functorially encoded into the category of categories: a ring R is encoded by the category of (left/right) linear representation of R (respec. $R - \text{Mod}$, $\text{Mod} - R$). In the same vein, it is proposed a encoding of a general propositional logic by a diagram of categories and functors given by the quasivarieties canonically associated to the algebraizable logics (in the sense of [BP]) connected with the given propositional logic. \square

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